Bright compact breathers

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In this communication we will consider the potential of some general classes of nonlinear lattice models to support bright discrete compact breather solutions (compactlets). We analyze the conditions for which such solutions are possible and classify the models as belonging in three general categories: a class with no compact breather solutions, one with one-parameter families of solutions, and a class with "isolated" solutions (i.e., no free parameters). In the latter two cases we construct the solutions and analyze their linear stability. The drastically different stability features of these solutions in comparison with their smoothly decaying counterparts are discussed. Stable breather solutions with compact support are identified in the one-parameter families of solutions, while the corresponding solutions found in the zero-parameter families are always found to be unstable.

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I. INTRODUCTION

The fact that nonlinearity can lead to the compression of wave packets is well known. When the action of the nonlinearity is balanced by that of dispersion, which tends to spread out a pulse, stable localized pulses, named solitons (or solitary waves more generally) are created [1]. This phenomenon is observed not only in continuum models, but also in discrete systems. As was first reported in [2], and subsequently studied by many authors [3] the balance between nonlinearity and dispersion leads to the creation of *lattice envelope solitons*. These entities, however, appear when the nonlinearity is small enough: then the amplitude plays the role of the small parameter of the problem. Envelope solitons can be mobile and extend over tens or even hundreds of sites. The region of localization decreases when the amplitude of the excitation increases. When the intensity (i.e., the square of the amplitude) of the excitation is still small, but the amplitude of the soliton is already comparable to unity (in dimensionless units), moving solitary waves which have a higher degree of localization compared with the envelope solitons [4] can be obtained. These are described by the discrete Hirota equation. If the amplitude of a single site becomes large enough, such that linear intersite interactions are much weaker than nonlinear self-action, strongly localized excitations (i.e., localized on very few sites) can exist [5,6]. These are called *intrinsic localized modes* (ILM's) and in the last decade they have received a large amount of attention. Like envelope solitons, ILM's appear as generic, robust solutions of nonlinear lattice equations. In addition, these modes, which are exponentially localized in space and temporally periodic (which is why they are also called *discrete breathers*), are of particular interest due to their ability to localize energy. As a result they have been theoretically proposed as the relevant mechanism for many physical phenomena. Already a number of review papers have appeared [7] that summarize these exciting recent developments.

On the other hand, a class of continuum solitary waves recently discovered is the one of solutions with compact support often referred to as *compactons* [8]. These solutions, contrary to what is the case for regular localized modes, have nonzero values only in a neighborhood of the real line and are strictly zero everywhere else. In particular, in many cases of interest they behave like a power of a trigonometric function inside their domain of nonzero values. This clearly contrasts with the exponential localization properties of regular ILM's and of envelope solitons.

A question that then naturally arises concerns whether discreteness can preserve solutions with compact support. In particular, it is of interest to examine whether a different class of breathers with compact support can be present in discrete setups. Only a few authors have considered this question to our knowledge. In [9], for Klein-Gordon chains, a compacton solution was found for the continuum analog of the equation and quasicompactification of the breather solution was observed for the genuinely discrete problem, but the numerical experiments were not conclusive. More recently, some case examples of discrete compactly supported breathers were considered in Fermi-Pasta-Ulam (FPU) chains [10]. For defocusing nonlinearities, in discrete nonlinear Schrödinger (DNLS) type contexts the recent work of [11] demonstrated the existence of such compact breathers for a special class of models. But a more systematic understanding of the nature and classes of possible solutions and the conditions for their existence and stability is still lacking. Furthermore, we should note that for some classes of continuum type models, all compacton solutions have been argued to be stable [12].

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From the above exposition, it can be clearly seen that a more careful examination of the possibility of formation of discrete compact breathers would clearly be desirable. We will hereafter call such solutions compactlets to highlight their genuinely discrete nature (see below). In exploring these topics, we will essentially follow a reverse engineering approach. That is to say that we will, in a form of experimental mathematics, consider a general class of models without focusing concretely on its physical motivation. This is an approach often used in continuum studies of compactons [8,13]. Here, we will study a benchmark system of DNLS type equations (i.e., discrete Schrödinger type equations with a nonlinear term and sharing the monochromatic gauge invariance of the regular NLS equation) for rather general classes of nonlinearities.

Our presentation will be structured as follows. In Sec. II, we will present the classes of models of interest and analytically obtain some relevant conclusions on the existence of breather solutions with compact support. We will also categorize general nonlinearities on the basis of this analysis. In Sec. III, we will numerically construct such compactlet solutions and analyze their linear stability and dynamics. Finally, in Sec. IV, we will summarize our findings and conclude.

II. ANALYTICAL CONSIDERATIONS

The general class of dynamical lattice models that we will consider will be of the form

$$i\dot{\psi}_n = -\Delta_2 \psi_n + f(\psi_{n-1}, \psi_n, \psi_{n+1}), \tag{1}$$

where $\Delta_2 \psi_n = C(\psi_{n+1}+\psi_{n-1}-2\psi_n)$ is the discrete Laplacian, C is a real constant (the so-called coupling constant), and the nonlinearity of a rather general type is restricted by the symmetry $f(\psi_{n-1},\psi_n,\psi_{n+1}) = f(\psi_{n+1},\psi_n,\psi_{n-1})$ and by the phase invariance that $f(\exp(i\Lambda t)u_{n-1}, \exp(i\Lambda t)u_n, \exp(i\Lambda t)u_{n+1}) = \exp(i\Lambda t) f(u_{n-1},u_n,u_{n+1})$ (Λ being real).

Notice that we will consider only nearest neighbor interactions in this work. Furthermore, we consider nonlinearities that are symmetric with respect to their inclusion of the left and right neighbors. This is, of course, not necessary for our general consideration, but considerably simplifies the exposition as we need to treat only one "end" of the compactly supported breather structure in what follows.

The approximation of the nearest neighbors allows us to introduce a definition of an N-site compacton as a lattice excitation such that $u_{n_0+j}\equiv 0$ for $j \leqslant -1$ and $j \geqslant N$ and $u_{n_0+j}\neq 0$ for $j=0,1,\ldots,N-1$.

We will set $C=1$ and vary the frequency ω of the excitation (one can easily show that one of these two quantities can always be scaled to a unit value). Finally, an important observation concerns the signs of the linear and nonlinear terms in the right hand side of Eq. (1). The opposite signs denote that we will have in mind defocusing nonlinearities for the creation of bright compactlet solutions. It should also be noted that the prototypical functional forms that one has in mind when writing the generalized Eq. (1) are $f(x,y,z) = |y|^2 y$ (in the case of the regular DNLS equation [7]) and $f(x,y,z) = |y|^2(x+z)$ for the well-known integrable variant of the DNLS equation, namely the Ablowitz-Ladik NLS (AL-NLS) [14].

For monochromatic solutions of the form $\psi_n(t) = \exp(-i\omega t)u_n$, we obtain the stationary problem

$$\omega u_n + (u_{n+1}+u_{n-1}-2u_n) = f(u_{n-1},u_n,u_{n+1}). \tag{2}$$

To identify compactly supported breather solutions in the lattice setup of the class of Eqs. (1) and (2), we argue that it is crucial to consider the u_{n_0-1}th site (if $u_{n_0}\neq 0$) for which the field is exactly zero. It is obvious that for existence of a compactlet one must require $f(0,0,0)=0$ in Eq. (2). This indicates that the solutions considered herein will be particular to discrete systems (hence the name compactlets).

The equation for the site $n=n_0-1$, for which $u_{n_0-1}=0$, will read

$$u_{n_0} = f(u_{n_0},0,0). \tag{3}$$

Even though this equation is rather trivial to obtain, it has significant implications that we should now examine. First, this equation suggests that it is impossible to support bright compactlets for an on-site substrate potential of the form $V(u_n,u_n^\star)$ (the asterisk denotes complex conjugation). For such a potential Eq. (3) will directly yield $u_{n_0}=0$ and all subsequent ordinates will also vanish. Hence the regular DNLS equation will *not* support such compactly supported structures. But it is easy to observe that neither will the AL-NLS equation, given its form of f. One then wonders whether there are generalized forms of f that could satisfy this equation with solutions other than $u_{n_0}=0$. A general class of such nonlinearities is

$$f(x,y,z) = g(|y|^2)(x+z) \tag{4}$$

for which $g(0)\neq 0$. A simple but rather general example of that form is given by

$$g(s) = \frac{A+Bs^k}{A+Cs^l}. \tag{5}$$

It should be observed here that for the purposes of g of Eq. (5) it is important that the constant factors in the numerator and the denominator are the same (and different from zero). Notice that if that is not so then Eq. (3) still yields $u_{n_0}=0$. More generally, for f's which are linear in their inclusion of the nearest neighbors, it should be true that $g(0)=1$. In that case, Eq. (3) becomes an identity and hence no constraints are imposed on the selection of the compactlet parameters by this equation. This is crucial and will be contrasted with the class of models given below.

The condition $g(0)=1$ requires the absence of linear dispersion. Indeed, let us consider the weakly nonlinear limit where $g(u_n^2)\approx 1+g'(0)u_n^2$. Then model (4) can be rewritten as $(\omega-2) = g'(0)u_n(u_{n-1}+u_{n+1})$. A direct consequence of this formula is that in the small amplitude limit compactlets can be either one site or two site. The one-site compactlet represents a trivial case. To consider the case of a two-site compactlet we observe that for the first (from the left) excited site n_0 we have $(\omega-2)=g'(0)u_{n_0}u_{n_0+1}$. Then, considering the site n_0+1 we obtain $g'(0)u_{n_0+1}u_{n_0+2}=(\omega-2)-g'(0)u_{n_0}u_{n_0+1}=0$, i.e., $u_{n_0+2}=0$. The provided analysis led us to the conjecture that for existence of an M-site compactlet with $M\geqslant 3$ there exists a threshold amplitude. We will give numerical evidence in support of this conjecture in Sec. III.

For linear coupling of the nonlinear term with its nearest neighbors (i.e., for f's linearly dependent on x,z), Eq. (3) can only amount to an identity when looking for solutions with compact support. However, it is not necessary for f to depend linearly on (x,z). An example of this type can be found in [11,15] with

$$f(\psi_{n-1},\psi_n,\psi_{n+1})=A\frac{(\psi_{n+1}^2+\psi_n^2)\psi_{n+1}^\star}{1+|\psi_{n+1}|^2}+\eta\psi_n\frac{|\psi_{n+1}|^2}{1+|\psi_{n+1}|^2}+(n+1\to n-1,n), \tag{6}$$

where the last parenthesis denotes the same expression but with the relevant change of indices. An alternative example of this type discussed in [11,16] reads

$$f(\psi_{n-1},\psi_n,\psi_{n+1})=A\frac{(\psi_{n+1}^2+\psi_n^2)\psi_{n+1}^\star}{1+|\psi_{n+1}|^2}+\eta\psi_n\frac{1-|\psi_{n+1}|^2}{1+|\psi_{n+1}|^2}+(n+1\to n-1,n). \tag{7}$$

In both models the factor A in front of the first fraction was absent in [15,16]. Its role in our exposition will be evident in what follows.

It is straightforward to see that in this case Eq. (3) will (generically at least) no longer be an identity. On the contrary, it will be an equation that determines u_{n_0}. In particular in the case of both models discussed above, Eq. (1) yields

$$|u_{n_0}|^2=\frac{1}{A-1}, \tag{8}$$

which reveals that compactlet solutions will be present only for $A>1$. In contrast to the case of Eq. (4), here we do not have a linear limit.

In view of the above results, we can classify general nonlinear lattice equations of the form of Eq. (1) as follows.

(1) The class of equations with no compactlet solutions. A subset of this set is the one with on-site nonlinearities (including the DNLS equation) as well as the set of linear nearest neighbor couplings such that $f(x,0,z)\neq(x+z)$ (including the AL-NLS equation).

(2) The class of models with one-parameter families of compactlets: since Eq. (3) becomes an identity, in the case of an M-site compactlet there will always be $M-1$ equations for M unknowns ($M-1$ site ordinates and the frequency of the compactly supported breather). These will generically admit one-parameter families of solutions. The nonlinearity of Eq. (4) belongs to this type if $g(0)=1$.

(3) Finally, the class of solutions with no free parameters, where Eq. (3) determines u_{n_0} and the remaining set of M equations will determine the ordinates of $M-1$ sites and the frequency of the compactly supported M-site breather. The models of Eqs. (6) and (7) belong to this category for $A>1$.

Almost all of the above results have been obtained by a careful consideration of Eq. (3).

Now, let us give some case examples.

For nonlinearities of the form of Eq. (4), for a one-site compactlet, u_{n_0} can be arbitrary and $\omega=2$. More generally for f satisfying identically Eq. (3), u_{n_0} is arbitrary and $\omega=2+f(0,u_{n_0},0)/u_{n_0}$. In the same case, for two-site bright compactlets of the form $\psi_n=\exp(-i\omega t)\times(\dots,0,u_{n_0},u_{n_0},0,\dots)$, u_{n_0} will be arbitrary and $\omega=1+g(|u_{n_0}|^2)$ for g of Eq. (5), while for general f's $\omega=1+f(u_{n_0},u_{n_0},0)/u_{n_0}$. Finally, for three sites $\psi_n=\exp(-i\omega t)\times(\dots,0,u_0,u_1,u_0,0,\dots)$, the relevant equations become

$$\omega-2=\frac{f(u_0,u_1,u_0)-2u_0}{u_1}, \tag{9}$$

$$\omega-2=\frac{f(u_1,u_0,0)-u_1}{u_0}. \tag{10}$$

We give the above equations to demonstrate the point that the equations will result in solvability conditions for each site in terms of u_{n_0}.

More generally, following what is known for regular ILM's from [5,6], there will be two main types of discrete compactlet, one that is centered on a site $n_c=N$ and one that is centered between sites $n_c=N+1/2$ (the subscript c for center). In the former case, the central site equation will read

$$\omega-2=\frac{f(u_{N-1},u_N,u_{N-1})-2u_{N-1}}{u_N}, \tag{11}$$

while in the latter it will be

$$\omega-2=\frac{f(u_{N-1},u_N,u_N)-(u_{N-1}+u_N)}{u_N}. \tag{12}$$

Similar considerations/calculations can be used for the models in which the coupling to the nearest neighbors is nonlinear as in Eqs. (6) and (7) above. In the latter case, for example, in the case of a two-site compactlet of the form $\psi_n=\exp(-i\omega t)(\dots,0,u_{n_0},u_{n_0},0,\dots)$, the frequency is given by

$$\omega-1=2+\frac{\eta}{A}, \tag{13}$$

$$\omega-1=2+\eta\frac{2-A}{A} \tag{14}$$

in the two models, respectively.

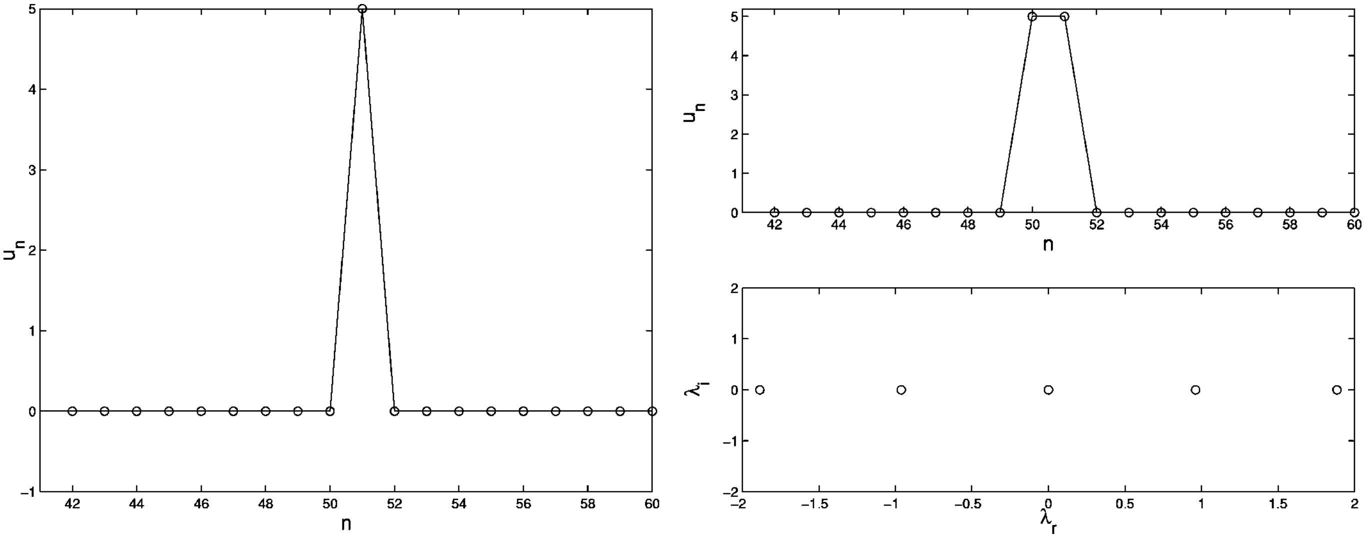

FIG. 1. The left panel shows a one-site discrete compactlet. $\omega=2$, $u_{n_0}=5$. Unless otherwise stated, the results are $g(|u_n|^2)=1/(1+|u_n|^2)$. The right panel shows the two-site solution (and its linear stability) for $\omega=1.038$ and $u_{n_0}=5$. The linear stability analysis shows the complex plane (λ_r,λ_i). The subscripts denote the real and imaginary parts of the eigenfrequency λ. Since there are no eigenfrequencies with nonzero imaginary part, the configuration will be stable.

Finally, a note of caution is in order. The technique of continuation of ILM's from the anticontinuum limit [7,17] has been very popular for ILM's in recent years. In particular, in many cases (see, e.g., [18]) the fact that ILM's are very strongly localized for weak couplings between the nonlinear oscillators has been exploited in considering them as consisting of only a few sites and studying their stability and dynamics in this few degree of freedom approximation. There should be no confusion between such modes and the ones presented herein. In our work, the modes are exact solutions of the stationary equations of interest and there is no approximation or weak coupling limit. Furthermore, the modes considered here are distinctively different from regular ILM's. The latter plotted on a semilogarithmic scale for NLS type equations can be clearly seen to have an exponential tail. On the contrary, the classes of discrete breathers considered here have strictly zero ordinates beyond the region of their support.

III. NUMERICAL RESULTS

We now turn to numerical experiments to investigate compactlet solutions and their stability. In particular, in order to create these modes, we solve Eq. (2), but not for an infinite lattice. We rather solve it only for the sites $n=1,\ldots,N$, when the discrete compact breather is centered either at site $n=N$ [the case of a $(2N-1)$-site compactlet] or at $n=N+1/2$ (the case of a $2N$-site compactlet).

While studying the stability of an M-site compactlet it is natural to consider perturbations of the whole chain. In what follows, however, we show that the small excitations of the sites which initially had zero displacements can be decoupled from the excitations of their neighbors. This immediately leads us to the conclusion that one-site and two-site compactlets are stable for the class of models with one-parameter families of solutions, as they can exist for any given amplitude. In Fig. 1 these results are illustrated by direct numerical computation of the solutions and their stability.

In order to study M-site compactlets with $M\geqslant 3$ ($M=2N$ or $M=2N-1$), we prescribe u_N for the class of equations with one free parameter. For $n=N$, we use the appropriate equation of Eqs. (11) and (12), depending on whether we are interested on a mode centered on a site or centered between sites, and solve the resulting N equations for $u_1,\ldots,u_{N-1}$ and ω. We then form the full solution (by symmetrizing the solution for $n>N$) and perform linear stability analysis. In particular, if f is given by Eq. (4), then using $\psi_n=\exp(-i\omega t)(u_n^{(0)}+\epsilon u_n^{(1)})$, we obtain the linearization equation for $u_n^{(1)}$

$$i\dot{u}_n^{(1)}=-\omega u_n^{(1)}-\Delta_2 u_n^{(1)}+g(|u_n^{(0)}|^2)(u_{n+1}^{(1)}+u_{n-1}^{(1)})+g'(|u_n^{(0)}|^2)(u_{n+1}^{(0)}+u_{n-1}^{(0)})\times(u_n^{(0)}u_n^{(1)\star}+u_n^{(0)\star}u_n^{(1)}). \quad (15)$$

It is a direct consequence of this equation that $u_n^{(1)}\propto\exp(i\lambda_\pm t)$ where $\lambda_\pm=\pm(\omega-2)$ for all $n<n_0$ and $n>n_0+M-1$; these represent stable excitations and the respective pair of eigenfrequencies will be present in the stability analysis of any such compactlet. This also means that one can restrict consideration to excitations of merely the sites that are "participating" in the original discrete compactly supported breather. Another general statement about the stability analysis of a compactlet is that due to the phase invariance (of the classes of models considered here) one will necessarily have two zero eigenvalues.

Let us now consider in more detail the stability of a three-site compactlet $(\ldots,0,u_0,u_1,u_0,0,\ldots)$, situated on the

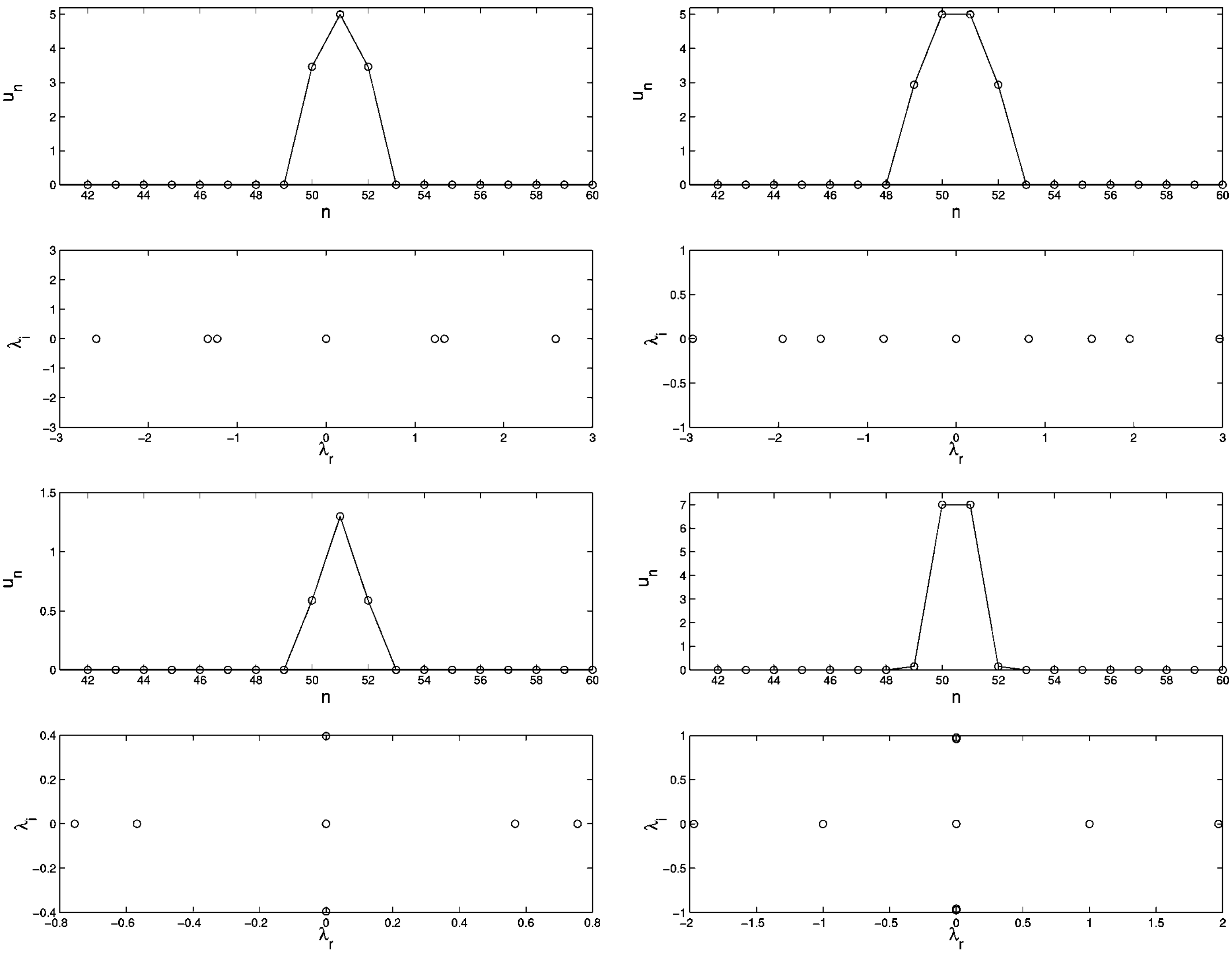

FIG. 2. Stable (left panels) and unstable (right panels) three-site (top panels) and four-site (bottom panels) solutions. For the top left panel $\omega=0.6677$ and $u_{n_0}=5$, while for the top right $\omega=1.4323$ and $u_{n_0}=1.3$. For the bottom left $\omega=0.4739$ and $u_{n_0}=5$, while for the bottom right $\omega=0.9996$ and $u_{n_0}=7$. The linear stability analysis in the case of the left panel indicates stability, whereas the imaginary eigenfrequencies in the case of the right panel show that the latter configurations are unstable.

sites $n=-1,0,1$. To this end, we rewrite Eq. (15) in terms of the renormalized displacements $w_{\pm 1}=u_1 u_{\pm 1}^{(1)}$ and $w_0=u_0 u_0^{(1)}$:

$$i\dot{w}_{\pm 1}=w_0-w_{\pm 1}+a_0(w_{\pm 1}+w_{\pm 1}^{\star}), \tag{16}$$

$$i\dot{w}_0=-w_0+\tfrac{1}{2}(w_1+w_{-1})+2a_1(w_0+w_0^{\star}), \tag{17}$$

where the overdot stands for the derivative with respect to $(\omega-2)t$ and

$$a_j=\frac{g'(u_j^2)u_0u_1}{\omega-2}=\frac{g'(u_j^2)u_0^2}{g(u_0^2)-1} \tag{18}$$

where $j=0,1$.
is positive for a decaying function $g(x)$. One can now consider the eigenfunction transformation $\{w_j,w_j^{\star}\}\rightarrow\{w_j^{\star}+w_j,w_j^{\star}-w_j\}$ Then, seeking solutions proportional to $\exp(i\lambda t')$, where $t'=(\omega-2)t$, we obtain the eigenfrequencies

$$\lambda_{\pm}^{(0)}=0,\quad \lambda_{\pm}^{(1)}=\pm\sqrt{1-2a_0},\quad \lambda_{\pm}^{(2)}=\pm\sqrt{4-2a_0-4a_1}. \tag{19}$$

These need to be multiplied by $\omega-2$ to be converted into eigenfrequencies of the original eigenvalue problem of Eq. (15). Considering the whole chain one has to add $\lambda_{\pm}$ to obtain the complete set of eigenfrequencies.

It follows from Eqs. (18) and (19) that three-site compactlets can be stable or unstable, depending on the ordinates of the sites. This conclusion has also been confirmed numerically (see, e.g., the two top panels of Fig. 2).

In a more general case, using $u_n^{(1)}=a_n\exp(-i\lambda t)+b_n\exp(i\lambda^{\star}t)$, we obtain an eigenvalue problem for

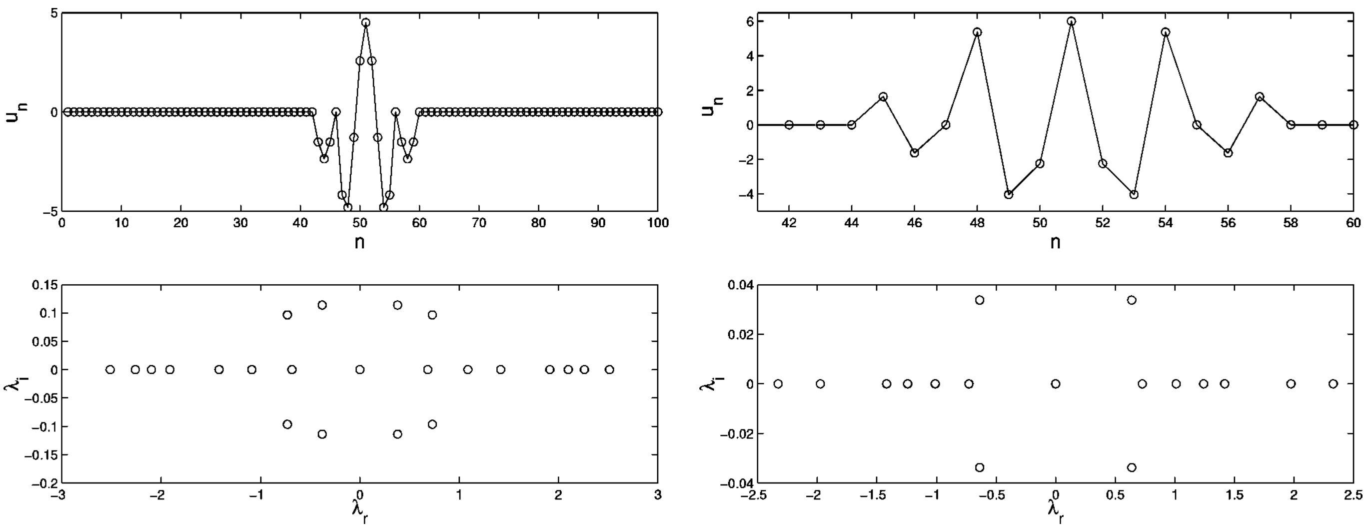

FIG. 3. The figure shows more exotic but highly unstable compactlet configurations. The left panel shows a compactlet with $\omega=0.9117$ and $u_{n_0}=4.5$. One can observe two complex eigenfrequency quartets due to the presence of two oscillatory instabilities. The relevant quartet eigenfrequencies have $(\lambda_r,\lambda_i)=(\pm 0.7289,\pm 0.0965)$ and $(\lambda_r,\lambda_i)=(\pm 0.3800,\pm 0.1139)$. Another elaborate spatial structure is shown in the right panel with $\omega=2.7279$ and $u_{n_0}=6$. The relevant quartet of eigenfrequencies is $(\lambda_r,\lambda_i)=(\pm 0.6379,\pm 0.0337)$.

$\{\lambda,\{a_n,b_n^\star\}\}$, which is subsequently solved numerically to identify the stability of the configurations. In the computations presented here,

$$g(|u_n|^2)=\frac{1}{1+|u_n|^2} \tag{20}$$

was used, but the main conclusions were also verified for $g=(1-|u_n|^2)/(1+|u_n|^2)$.

In particular, as a result of the numerical computations for the one-parameter families of compact discrete breathers, it was found that four-site compactlets, similarly to three-site ones, can be either stable or unstable (see, e.g., the two bottom panels of Fig. 2). However, gradually, as the size of the configurations grows, the range of stability of multi-site configurations is quite small and most configurations found for $N\geqslant 3$ (i.e., with five or more sites) were most often unstable in the cases examined; however, some stable configurations were found, for instance from the concatenation of smaller stable (e.g., three-site) building blocks. An example of more exotic (but unstable) compactlets is shown in Fig. 3. Shown also are the corresponding instabilities, which do not always stem from imaginary eigenfrequencies but can also be oscillatory, giving rise to Hamiltonian Hopf bifurcations [19].

A significant difference should be highlighted between these solutions and the ones known for regular DNLS type equations. For the latter, the site-centered mode [5] is well known to be stable, while the intersite-centered [6] mode is known to be unstable (see, e.g., [7]). In the case of compactlets, both solutions appear to be stable. This seems to be in partial agreement with the conclusions of [12] about different classes of compactons being stable. On the other hand, in the discrete case, contrary to the continuum observations of [12], for a larger number of sites, most solutions are unstable, particularly so for compactlets involving five or more sites. For three and four sites, we were able to identify both stable and unstable solutions depending on the values of the ordinates. This is also not common to DNLS type equations (at least in $1+1$ dimensions). For instance, a single pulselike ILM is always stable if it is centered on a site and always unstable if centered between sites. This conclusion does not depend on the ordinates of the mode's lattice sites. Furthermore, the spectrum of the compactlets does not include the spectrum of the background state on which they exist. This is in sharp contrast with the well-known results for regular ILM's. In particular, in the case of a DNLS equation in $1+1$ dimensions, the standing wave pulselike solutions of the focusing case, or the kinklike solutions of the defocusing case (dark solitary waves), encompass the so-called phonon band or continuous spectrum. The continuous spectrum consists of the extended wave excitations permissible by the uniform steady state on which the ILM exists. This continuous spectrum alongside the point spectrum of localized (L^2) eigenfunctions constitutes the full linearization spectrum. However, this is not true for the compactlets. In the latter case, the decoupling (see above) of perturbations along the compactlet sites and those along the rest of the lattice sites essentially causes the "collapse" of the continuous spectrum to a single eigenfrequency ($\lambda_\pm$). Finally, oscillatory instabilities do occur for bright compactlets as they do for their regular ILM counterparts, provided that π phase differences appear in the coherent structure (see, e.g., [20]).

The time evolution of compactlet instabilities was probed by means of numerical time integration (using explicit fourth and eighth order Runge-Kutta integrators). It was found that in the case of three or four sites where there are structurally similar configurations which are stable, the compactlet would not be structurally destroyed but would rather oscillate and thus approach such stable configurations. Highly

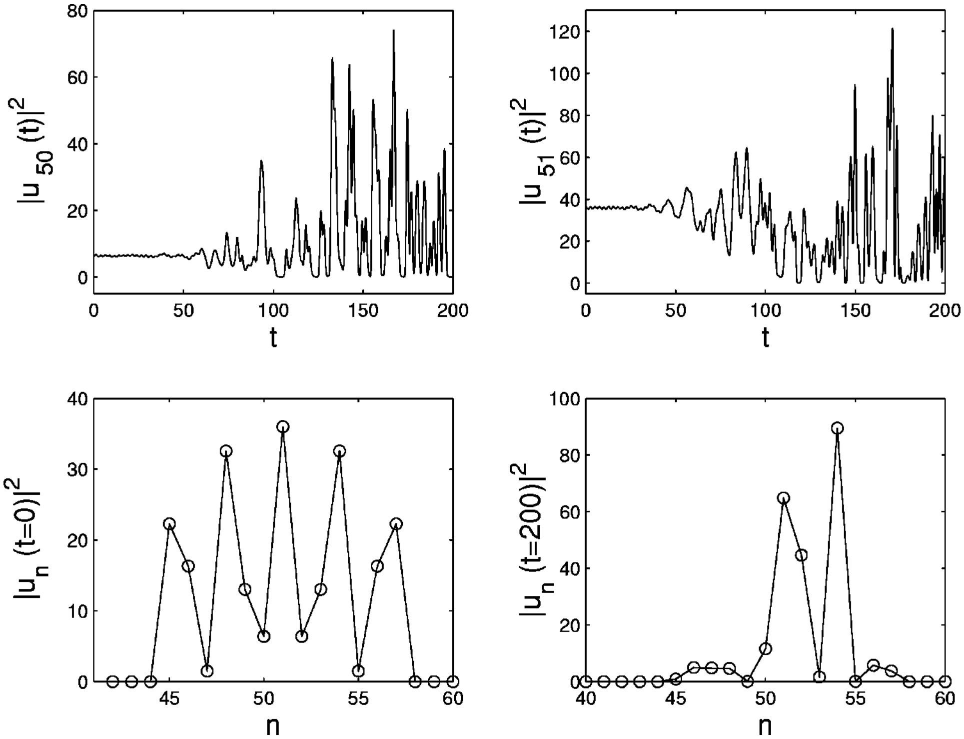

FIG. 4. The time evolution of the configuration of the right panel in Fig. 3. The top left and right panels show the time evolution of the two central-most sites of the compactlet. The two bottom panels show the snapshots of the spatial profile of the configuration at $t=0$ (the initial condition of the exact compactlet plus a small perturbation along the unstable eigendirection) in the left panel and the configuration at $t=200$ (right panel). $t=200$ was the duration of this simulation, but we also performed numerical experiments up to $t=2000$, observing that the compactlet gradually breaks up into smaller stable entities (predominantly single site compactlets). It is interesting to note that no visible signs of extended wave radiation traveling toward the boundaries have been observed in these runs.

unstable configurations with a large number of sites were found to be destroyed. Such an example occurs in the case of the oscillatory instability of one of the structures shown in Fig. 3. The oscillatory instability [being manifested in a quartet of eigenfrequencies $\lambda=(\pm 0.6379,\pm 0.0337)$] is seen to destroy the discrete compactlet through oscillations of increasing amplitude in Fig. 4. It is, however, noteworthy, that no significant extended wave radiation appears to be present in the simulation.

To complete the consideration of one-parameter families of compactlets we display in Fig. 5 the dependence of the displacements of the side atoms of a five-site compactlet versus the displacement of the middle atom (which determines the amplitude of the compactlet). It was conjectured in the previous section that the amplitude of the five-site compactlet has an existence threshold u_{th}; namely, at $u_3=u_{th}^{(1)} \approx 3.73$ the displacement of the first atom becomes identically zero (also the second atom's displacement "jumps" to a lower value). At this amplitude value the five-site compactlet degenerates into a three-site compactlet. That is to say, for lower amplitude values only three sites have nonvanishing ordinates and hence five-site compactlets can exist only for $u_3>u_{th}^{(1)}$. Subsequent decrease of the central site amplitude results, for $u_3=u_{th}^{(2)}\approx 1$, in the transformation of the three-site compactlet into a one-site one (and thus three-site compactlets will exist for $u_3>u_{th}^{(2)}$). These transitions can be observed in Fig. 5 and are naturally reflected in the dependence of the frequency of the compactlet on its amplitude [see Eq. (11)]: it has two "jumps" at $u_{th}^{(1)}$ and $u_{th}^{(2)}$.

Finally, from the class of models with no free parameters, the model of Eq. (6) was examined and since the linearization problem is extremely tedious, for simplicity the case with $\eta=0$ was considered. However, in the latter case all the solutions identified were found to be unstable (and hence are not shown). Some intuition about this instability (even though this is not a proof) may be obtained on the basis of lack of free parameters in the configuration. Once a perturbation is performed there is no other configuration of this type to reshape into (i.e., no other "fixed point," in the space of configurations, of the same type in the vicinity of the original solution) and hence it *may* be more natural to expect in this case that the configuration will be destroyed.

IV. DISCUSSION

In this work we have discussed the possibility of formation of compactly supported, breathing in time coherent structures in discrete models with nearest neighbor interactions (both linear and nonlinear). The compactlet solutions considered here are genuinely discrete and have no continuum analog. A number of conclusions have been inferred from the equation for the first site beyond the compactly

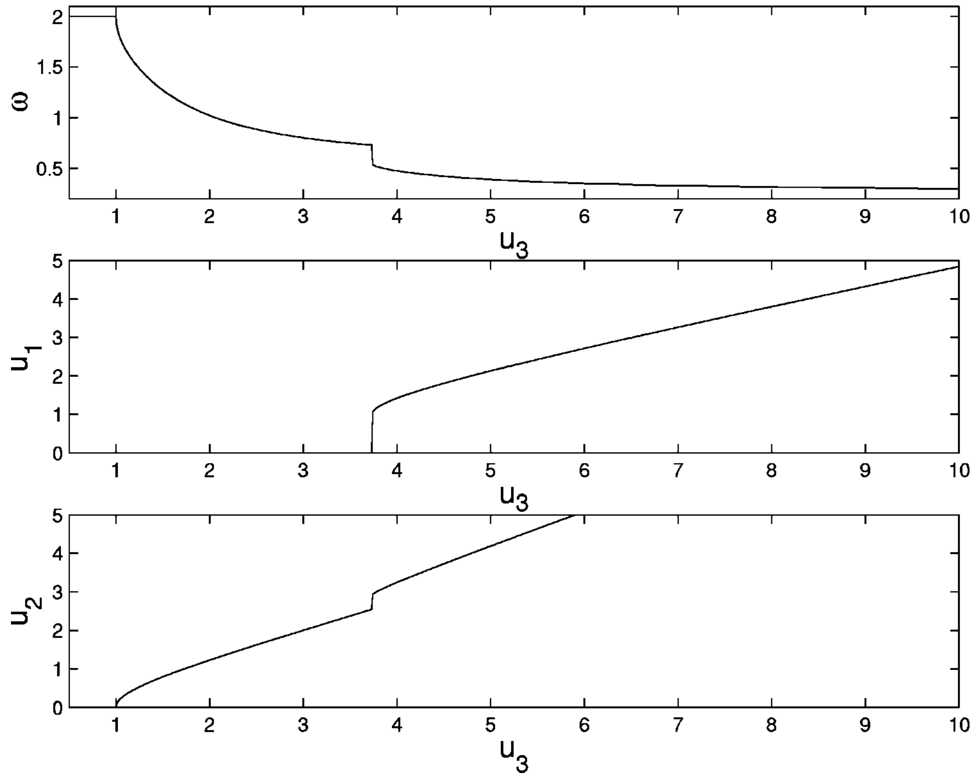

FIG. 5. The top panel of the figure shows for a five-site compactlet the dependence of the frequency as a function of the displacement of the middle (i.e., $n=3$) site. The middle panel shows the dependence of the first site u_1 as a function of u_3, and the bottom panel the dependence of u_2 on u_3. Notice, as we proceed from the right of the figure toward the left, the very clear transition from a five-site to a three-site compactlet at $u_3=u_{cr}^{(1)}\approx 3.73$, and the less abrupt but also discernible transition from the three-site compactlet to a one-site compactlet for $u_3=u_{cr}^{(2)}\approx 1$ (see also text).

supported solution. On the basis of this equation, we have provided a general classification of models to ones that cannot support compactlets, ones that have one-parameter families of such solutions, and ones that have zero-parameter families (i.e., isolated solutions) of that form. Many of the well-known models such as the DNLS and AL-NLS equations belong in the first class. We also specified and studied examples of the latter two categories. We analyzed the details of such compactlet solutions and calculated them explicitly in simple cases (of few lattice sites). We then proceeded to construct such solutions numerically. While in the case of no free parameter models we found even the simplest compactlets to be unstable, in the case of one free parameter, the single-site and two-site compactlets were always stable. It was found that for larger numbers of sites in these compact discrete breathers, they will be mostly unstable, even though in some cases (especially for three- and four-site compactlets) stable such solutions exist. We have conjectured that the one-parameter family of compactlets possesses a threshold amplitude below which an N-site ($N>2$) compactlet cannot exist. Numerical evidence in support of this conjecture for the cases of $N=3$ and $N=5$ has been provided.

Naturally, as this type of discrete breather is only starting to be explored, there are many questions that merit consideration and would be of interest to future studies. One of them concerns whether experimentally realizable physical models containing such modes can be identified and consequently whether such solutions can be observed. In elucidating some of the aspects of the nonlinearities permitting such solutions, we hope it will become easier to answer this question in the near future. Furthermore, understanding more details about the point spectrum of such compactlets may prove very significant in identifying their stability picture and unveiling their characteristics. Additional future directions involve the interaction of such coherent structures and their characteristics of motion in the discrete setup (i.e., whether they shed radiation waves or not). Research work in these areas is currently in progress and will be reported in future publications.

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